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SOLUTION OF THE APPROXIMATION PROBLEM
OF NETWORK SYNTHESIS
WITH AN ANALOG COMPUTER

BY
STANLEY LEHR

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ABSTRACT

Analytical solution of the approximation problem of network synthesis entails a large amount of computational work. This report describes an experimental method of solution by means of a computer based on the electrostatic analogy in which voltages in a conducting sheet, set up by line currents, represent network immittance functions. The resulting approximating function is described in terms of its pole and zero locations and a multiplicative constant. The method of operation of the analog computer is simple. The poles and zeros are moved around in the conducting medium until the desired immittance is obtained. The continuous scan feature of the computer makes possible visualization of the response and, what is more important, immediately shows the effect of moving any of the poles or zeros. A convergence procedure for the perturbation of these critical points has been developed so that an approximation within the required accuracy is often possible within a very short period of time.

Approximation from the viewpoint of economization is carried out with the objective of reducing the number of critical points of a known approximating function in order to reduce the number of required elements in the network. The process entails the removal of a pole or zero and redistribution of the remaining critical points to return, as closely as possible, to the required response. The removal of one, or a pair, of critical points, perturbation of the remaining ones, and adjustment of the multiplicative constant is then repeated. The process is stopped just before the number of poles and zeros has been so greatly reduced as to make the approximation within allowed deviation unobtainable.

Positive real functions are guaranteed to give physically realizable driving-point immittances. To insure PRF results, the analog approximation is made to the real part of the driving-point immittance. The computer is well-suited for approximations to the real or imaginary parts, as well as the amplitude function.

Approximations were carried out to the transcendental function arising from the input admittance of a shorted transmission line. The synthesized networks were found to give smaller deviations than known partial fractions expansions. Furthermore, the analog gave networks with fewer elements.

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I. Introduction

Although the theory of synthesis of passive, linear, lumped-parameter networks is well-known, there is, as yet, no completely satisfactory solution to the approximation problem. Usually, a synthesis can be carried out after the immittance* function can be described by means of an algebraic expression of a specified form. Often, the graphical representation of the function is given and the problem then becomes one of finding the equation of the curve, in the required form, which will approximate the desired function to a required degree of accuracy. Analytical methods of solution exist, but the amount of computational work is generally very large.

This report describes an analog computer which provides an experimental means of solving the approximation problem. The procedure is essentially one of trial-and-error, but converges very rapidly. The solution is found directly in terms of the locations of the poles and zeros of the function representing the required response.

The methods used here to solve the approximation problem are made possible because of an analogy which exists between electrostatics and the type of rational analytical functions which arise in the network theory (References 1, 4, and 8). Thus, the highly developed mechanism of potential theory becomes available for the solution of circuit problems. The common and easily developed "feeling" for electrostatic problems may be applied to give useful qualitative results in network synthesis.

II. The Electrostatic Analogy

Any immittance function can be described in terms of its poles and zeros and a multiplicative constant,

$$F(p) = \frac{H(p - p_2)(p - p_4) \dots}{(p - p_1)(p - p_3) \dots} \quad (1)$$

where,

$p = \sigma + j\omega$ is the complex frequency variable

H is the multiplicative constant

p_2, p_4, \dots, p_{2n} are the zeros of $F(p)$

$p_1, p_3, \dots, p_{2n-1}$ are the poles of $F(p)$

The logarithm of the absolute value of the immittance is

* In this report, the general term "impedance function" includes impedance or admittance, of both driving-point or transfer type.

$$\ln|F(p)| = \ln H + \sum_n \ln|p - p_{2n}| - \sum_n \ln|p - p_{2n-1}| \quad (2)$$

and is called the gain function of the network. Note that the gain function, at any point p , is equal to the sum of the logarithm of the distances from p to the zeros, minus the sum of the logarithm of the distances to the poles, plus a constant.

Now, let us examine the voltages in a uniform, infinite, conducting sheet pierced by line current sources.

Consider a finite region of the infinite conducting plane of thickness δ , as shown in Fig. MRI-13293. O is the origin, where the current I is introduced into the plane. Q is any point whose distance from the origin is q , and K is a reference point located at distance k from O .

Since the conducting sheet is infinite, the current emanates radially from the source at O . But,

$$I = JA \quad (3)$$

The assumption is made that there is no current flow in the δ direction. Also,

$$R = \frac{\rho l}{A} \quad (4)$$

hence,

$$V = IR = J\rho l \quad (5)$$

where, J = current density

ρ = resistivity of paper

l = length of conducting path

A = area presented to the current

Then, at any radial distance r , the change in voltage dV for a differential change in radius dr is

$$dV = \frac{I\rho dr}{2\pi\delta r} \quad (6)$$

Integrating from K to Q results in

$$\begin{aligned} V_{KQ} &= \frac{I\rho}{2\pi\delta} \ln \frac{k}{q} \\ &= C - \frac{IR_0}{2\pi} \ln q \end{aligned} \quad (7)$$

where R_0 is the resistance per square of the paper and C is a constant.

It is seen, then, that for a constant current source, the voltage at any point is proportional to the logarithm of the distance to the current source.

If now, the conducting sheet represents the p -plane, and if positive currents are introduced at points corresponding to the poles and negative currents are introduced at points corresponding to the zeros, all currents being of the same magnitude, the resultant voltage becomes, by superposition,

$$V = A + \frac{IR_0}{2\pi} \left[\sum_n \ln|p-p_{2n}| - \sum_n \ln|p-p_{2n-1}| \right] \quad (8)$$

Eq. (8) is identical in form with Eq. (2) for the gain function of a network. The constant multiplier, $IR_0/2\pi$, determines the scale factor. The constant A determines the zero level of the voltage, i.e., the impedance level of the gain function.

III. The Logarithmic Transformation

Double-layer, circular, electrolytic tanks representing the p -plane have been used (References 2 and 3) as a tool in network analysis. The use of carbon impregnated paper as a conducting medium is facilitated by the logarithmic transformation (Reference 6)

$$\begin{aligned} W &= \ln p \\ &= \ln|p| + j\phi \\ &= U + jV \end{aligned}$$

$$\text{where, } p = |p|e^{j\phi} = \sigma + j\omega \quad (9)$$

The Riemann surfaces of the p -plane are all represented by the infinite W -plane sheet. The single Riemann surface

$$0 \leq \phi < 2\pi$$

made by taking the branch cut in the p -plane along the positive real axis, corresponds to a semi-infinite W -plane strip of width 2π . The transformation is shown in Fig. MRI-13294.

The advantages of the logarithmic transformation led to the construction of the physical analog in the logarithmic plane. These advantages are:

1. The logarithmic coordinates are convenient for many problems.
2. A sizeable range of frequencies can be represented in a convenient size sheet.
3. Calibration is simple using known, constant-slope response curves.
4. The accuracy is uniform over the useful region of the plane.
5. Errors due to the finite size of the plane are easily made negligible.

The main advantages of using the conducting paper, rather than an electrolyte, are:

1. The paper is not affected by shock, whereas disturbances of the electrolyte cause ripples and consequent distortion of the field.
2. Use of direct current is possible since there is no dissociation problem.
3. Paper is more conveniently handled than a fluid.

IV. Symmetry Conditions

A. Symmetry about the Real Axis

Since network immittances are analytic functions expressible as ratios of polynomials in p with real coefficients, poles and zeros appear either as complex conjugate pairs or on the real axis. This means that the W -plane will have current sources and sinks placed symmetrically about the π (real) axis. Thus, no current will flow across the π -axis and the plane may be cut along this line without disturbing the field, or voltage distribution. Only one half of the transformed plane is then used and it contains all the necessary information. Poles and zeros on the real axis are reduced to half their normal value since only half the current would normally flow into the half-plane used in the analog.

This symmetry condition is shown in Fig. MRI-13295-A.

B. Symmetry About the Imaginary Axis

Normally, the poles and zeros of an immittance are not symmetrically placed about the imaginary axis. If the symmetrical critical points are introduced into the analog, the voltage produced along the $\pi/2$ (imaginary) axis, that is, the gain, is simply doubled (Reference 5). This result can be proven mathematically in the following fashion (Reference 4): Let the original immittance (normalized) be

$$F(p) = \frac{\pi(p-p_{2n})}{\pi(p-p_{2m-1})} \quad (10)$$

as in Eq. (1). The logarithm of the magnitude of the immittance is then the voltage measured along the imaginary axis or,

$$V = \sum \ln \left| \frac{j\omega-p_{2n}}{j\omega-p_{2m-1}} \right| \quad (11)$$

With the addition of the image critical points, $F(p)$ is changed to

$$F_1(p) = \frac{\pi(p-p_{2n})(p-p_{2n}^*)}{\pi(p-p_{2m-1})(p-p_{2m-1}^*)} \quad (12)$$

Now the logarithm of the magnitude of this function along the imaginary axis is the measured voltage

$$\begin{aligned} V &= \sum \ln \left| \frac{j\omega-p_{2n}}{j\omega-p_{2m-1}} \right|^2 \\ &= 2 \sum \ln \left| \frac{j\omega-p_{2n}}{j\omega-p_{2m-1}} \right| \end{aligned} \quad (13)$$

The result shows that the magnitude measurement is simply doubled.

The imposed symmetry about the imaginary axis makes it possible to cut the analog plane in half again, as did the symmetry about the real axis. This time critical points on the imaginary axis are reduced to half-currents and only one quarter of the original p -plane is represented in the analog. This symmetry condition is shown in Fig. MRI-13295-B. Furthermore, the

imaginary axis now represents a stream line (i.e., no current flows across it). Since all current flow lines approach the imaginary axis asymptotically, and since current and voltage are conjugate functions in the analog, then the constant voltage lines are all perpendicular to the imaginary axis, which means that the voltage measuring probe does not have to be so accurately positioned.

V. Description of the Analog Computer

The computer is mounted on a standard size machinist's table having three shelves which, with the addition of an oscilloscope, makes a complete, easily portable unit. A drawing board mounted on the top shelf acts as the support for the conducting-paper plane. The remaining components are placed on the other two shelves.

Currents are introduced into the conducting plane by means of current probes. These probes are simply steel needles pressed into brass sleeves. The sleeves facilitate the attachment of leads. The current probes are fed from d-c sources which are constant voltage supplies of either plus or minus 150 volts. 150 kilohms is inserted in series with each probe for full currents and 300 kilohms for half-currents. Thus, the full currents are of the order of one milliamperes. This was chosen as a convenient value since it leads to convenient values of scanning voltage. In calibrating (see p. 10), the voltage gradient along the real frequency axis is approximately 0.6 volts per decade of frequency and is readily handled by the display system. Lower current sources would require further voltage amplification while higher currents tend to heat, and possibly burn, the conducting paper. Series potentiometers of 20 kilohms are used to adjust all currents to the required values. These are necessary since the impedance seen by a current probe depends on its position in the conducting sheet as well as on the total number of probes in the sheet. In order to preserve the drawing board, a replaceable cork sheet is placed beneath the conducting paper. The current probes are held in place by pressing into the cork sheet.

The voltage along the real frequency axis is scanned by a rolling voltage probe. The voltage probe comprises a ball bearing with a copper sleeve pressed on to the outside of the race. The sleeve is circular in cross-section and hence, point-contact is made with the paper. The voltage probe carriage is supported on a length of rectangular waveguide and is driven across the sheet by means of a motor and belt drive. Reversing micro-switches are mounted at each end of the guide. The driving motor is a commutator a-c type with a forced air cooling system. Since the total scanning time of the paper is short, a cyclic duty motor, instantly reversible every five seconds, is required. A high torque to inertia ratio and a cooling fan prevent overheating.

The voltage at the pick-up (scanning) probe is displayed on a Dumont 304H cathode ray oscilloscope. The scope's d-c amplifiers have sufficient gain

for the voltages encountered so that no pre-amplifiers are necessary. Horizontal deflection for the display system is obtained from a Helipot precision linear potentiometer driven from the voltage probe carriage by a continuous belt drive. A tension adjusting coil spring is inserted in the belt to minimize backlash. Drag on the probe carriage has been reduced to a minimum established by the friction of the pulleys and potentiometer bearings. Since the output from the analog plots a logarithmic gain function against the logarithm of frequency, the face of the cathode ray tube has been calibrated in log-log coordinates. This was done by spraying the tube face with an acrylic coating. A log-log grid was then inscribed with a needle-point on the surface thus formed. The scratch marks were filled with marking crayon to make permanent coordinates directly on the face of the tube. A long-persistence tube is used in order to retain the trace for as long a time as possible. This makes the effect of movement of critical points readily observable. The computer is thus a continuous scan mechanism. A picture of the analog computer (without the display oscilloscope) is shown in Fig. MRI-13296.

VI. Errors in the Computer

The percentage error depends, somewhat, on the particular problem involved. There are certain errors, however, which are inherent to the device. These are analyzed below. The principle sources of error are:

1. Nonuniformity of the conducting paper
2. Paper shift
3. Finite size of the plane
4. The probes
5. The oscilloscope
 - a. nonlinearity of the amplifiers
 - b. parallax
 - c. pickup

1. Nonuniformity of the Conducting Paper

Changes in the resistivity of the paper may be caused by a change in the spacing of the rollers while the paper is being made or by impurities and inhomogeneities in the material. Errors due to a change in roller spacing are completely negligible since the long dimension of the conducting sheet used in the analog is taken from the width of the roll and, usually, only 3.41

inches of length along the roll are needed. The latter cause of nonuniformity was investigated by measuring the resistance per square of the conducting sheet. A pair of coaxial circular conductors were placed in contact with the conducting medium. The resistance between these conductors is directly proportional to the resistance per square of the conducting sheet, the proportionality constant being determined by the geometry. The area between the conductors is approximately 1.3 square inches. Hence, this method is a measure of the average resistivity over a comparatively small area. Tests were carried out on a 5 foot sheet cut from the roll, which is about 31 inches wide. About 200 readings were taken, or one reading every three inches. The results showed that the paper has a resistance of 1800 ohms per square with variations of less than 0.4 percent.

2. Paper Shift

Paper shift was an extremely troublesome problem. In order to assure good contact of the rolling voltage probe, it is necessary to spring load the contacting wheel. Unless the conducting paper is securely held, the voltage probe will shift the paper as it moves across the plane. The spot on the cathode ray tube then does not retrace the same curve when the voltage probe scans in the opposite direction. Paper shift has been eliminated with the use of spring-fingers, at both ends of the conducting sheet, arranged so as to flatten the sheet and put it into tension.

3. Finite Size of the Plane

In the logarithmic plane no errors are introduced by the finite width of the paper, providing that the ratio of width to length is such that the conformality of the transformation is preserved. The conducting sheet was chosen 20 inches long as a convenient size. Four decades of frequency are used, making each decade five inches long. The corresponding width must represent a change of 2π radians. Hence, for the width x ,

$$\frac{x}{2\pi} = \frac{5}{\ln 10} \quad (14)$$

Therefore, x is 13.65 inches. But the analog represents only a quarter of the p -plane. In practice, then, the conducting sheet is 3.41 inches wide.

The transformation requires the use of a semi-infinite sheet. There is an error introduced in using a finite length to represent the logarithmic plane. Consider the field far removed from a finite region where all the poles and zeros are located. The return for any excess current introduced into the plane is taken at infinity. Then the equipotentials far from the furthest essential point are constant amplitude lines or lines perpendicular to the length of the conducting sheet. Equipotentials are painted across the sheet with silver paint to represent, at the two ends of the paper respectively, the origin

and the point at infinity. Analysis shows (Reference 4) that if the equipotential representing the point at infinity is placed at seven times the distance to the furthest singular point, the errors due to the finite size of the plane will never be greater than one percent. In the computer, this ratio has been increased to ten as an added precaution. The same situation occurs at the other end of the strip, and it is likewise satisfied by leaving an unused decade at the zero end. The sheet is four decades long. Thus, the two center decades are available for pole and zero location, or the frequency range is 100 to one.

4. The Probes

Because the current probes are finite in size they distort the field by disrupting the homogeneity of the conducting medium. Holes left in the paper after the removal of the needle probes have the same effect. The distortion of the field, however, is negligible at distance greater than ten times the radius of the probe (Reference 4). Since the probe radius is .0125 inches, distortion due to probes is not noticeable at distances of more than one-eighth inch. A similar condition holds for the voltage probe since the rolling wheel does not make actual point contact but has approximately a circular contact area of small radius. A further error is introduced if the voltage probe does not accurately follow the line of the real frequency axis. This error has been minimized in the computer by making the constant voltage lines perpendicular to the scanning axis (p. 6).

5. The Oscilloscope

Nonlinearity of the amplifiers can be kept to a minimum of 3 percent by proper adjustment of the scope and use of fresh tubes. The center three inches of the five inch tube used is almost perfectly linear; hence, nonlinearity is not an important error in the center of the tube.

Parallax error was minimized by inscribing the log-log coordinates directly on the face of the cathode ray tube (p. 7). The separation of the spot and grid is then only the thickness of the glass face of the tube.

Pickup produces error since it tends to give thickness to the spot, making fine focusing, and hence exact location of the point, impossible.

In most cases, the largest errors are introduced by the display system, i.e., the oscilloscope. Because of diffusion, nonlinearity and parallax, the spot cannot be located closer than within one-sixteenth of an inch. This introduces an error on the log-log grid of no more than five percent. The maximum overall error is then approximately six percent.

If the function in question has a critical point (zero or pole) within an eighth of an inch of the real frequency axis, the display is not valid immediately within the vicinity of the current probe. This can be

readily remedied, however, by calculating a point or two on the curve in question.

VII. Method of Operation

The method of operation of the computer is comparatively simple. The desired form of the gain function is drawn directly on the face of the scope with China marking crayon. The poles and zeros are then moved around in the conducting sheet until the picture on the cathode ray tube agrees with the desired one within the specified tolerance.

In this manner, the approximation problem of network synthesis can be solved in comparatively a very short time. In many cases, the corresponding analytical procedure would require several days. Furthermore, problems requiring a large number of essential points are handled with almost the same facility as the simple problems. This is not at all true of the analytical approach.

VIII. Calibration Procedure

Calibration of the log-amplitude scale presents no difficulty. Essentially the procedure involves setting up a known amplitude function in the analog and setting the gain controls to see the proper response on the cathode ray tube. The simplest response is that due to one pole (or one zero). If the pole is located at the origin, the response is a constant slope line. That is,

$$F(\omega) = \frac{1}{\omega} \quad (15)$$

From which,

$$\text{slope} = \frac{d(\log |F|)}{d(\log \omega)} = -1 \quad (16)$$

The procedure, then, is to set the horizontal positioning and gain controls so that the position of the voltage probe corresponds to the frequency scale, then scanning the output voltage from a simple pole at the origin and setting the vertical gain control so as to see a unit slope line. For convenience, this slope has been permanently inscribed on the face of the cathode ray tube. It should be noted that a simple pole at the origin corresponds to a current source of one-fourth normal value if only one quadrant of the p-plane is represented by the analog. If 1/2 the p-plane is used, the calibrating singularity will have half the normal value of current.

No mention has, as yet, been made of setting the vertical control. It will be recalled that one of the variables of the approximations is the determination of the multiplicative constant of the immittance (Eq. (1)). When the logarithm of the amplitude is taken, this becomes an additive constant (Eq. (2)). But the addition to a constant of a curve means simply raising or lowering the position of the curve. Thus, the vertical position control is a multiplicative adjustment in the approximation procedure.

IX. The Approximation from the Viewpoint of Economization

If the approximation has its start in an arbitrary choice of the number of poles and zeros and a random location of these critical points, the problem becomes a question of relocation of these points to obtain the best approximation and the convergence technique is not readily apparent. In fact, the solution of this problem involves so much ingenuity that it may at times seem to be more of an art than a science. For this reason, the approximations carried out in this paper start with a known approximating function. An economization process is used to reduce the number of poles and zeros as much as possible, thus minimizing the number of elements and obtaining the best approximation for this situation. The method uses several of the critical points of the known approximating function as a starting point. Enough poles and zeros are chosen to give a good approximation in the range of interest. Then, the pole or zero having the least effect on the response is removed and the remaining critical points are perturbed until the response is returned, as closely as possible, to that which existed before the removal of the singularity. The process of removal of one, or a pair, of critical points and perturbation of the remaining ones is then repeated. Usually, the amount of rearrangement necessary after the removal of a pole or zero is not too great until a minimum number of critical points is reached. Further economization makes approximation within specifications impossible. At this point, the optimum synthesis is reached. Analytic economization, sometimes called "quantization", has also been used (References 5 and 12) to approximate continuous pole and zero distributions with a finite number of critical points.

X. The Perturbation Technique

Although it is not always possible to predict the movement of critical points in order to converge to the required approximation, a limited perturbation technique has been evolved. An analytic approximation technique, by successive adjustments specified in terms of shifts of pole and zero positions made to reduce deviation from desired characteristics, has been successfully carried out elsewhere (Reference 7). However, the methods are much more laborious than those described here and a check on optimum synthesis cannot readily be made.

The first question in the economization concerns itself with the order in which the poles and zeros are removed. In general, the critical points are eliminated as follows:

1. Points outside the region of interest.
2. Points lying at the far ends of the region of interest.
3. Points furthest from the real frequency axis.
4. Current sources of one polarity lying between sources of opposite polarity.
5. Other points, if possible.

Step 4 is perhaps most in need of explanation. It sometimes occurs that three critical points, say a zero between two poles, lie along a line approximately parallel to the scanning axis (constant phase line). It is then possible to economize by removing the center critical point. The success of the economization depends somewhat on the original positions of the critical points and on the total numbers of poles and zeros in the finite region of the plane. Another point of interest is the method of perturbation which leads to a most rapid convergence. An empirical technique, applicable in many situations, has been evolved. It should be said here that control of the multiplicative constant plays an important role in the convergence procedure. Vertical positioning is adjusted following every other step, that is, after each removal of a pole or zero; again after each perturbation of each remaining critical point.

The critical points may be classified according to their location in the logarithmic plane as:

1. Poles or zeros close to the real frequency axis. These points lie close to maxima or minima of the required curve or where the curve has a steep slope. The position of these points is most critical, therefore they are the first to be adjusted. Moving a pole closer to the scanning axis (along a constant amplitude line) increases the peak. Moving the pole parallel to the scanning axis (along a constant phase line) keeps the amplitude of the peak essentially constant but changes the frequency at which the peak occurs. The effect of small movement along a constant amplitude line is not as readily observable far from the critical frequency as is small movement along a constant phase line.
2. Poles or zeros far from the real frequency axis. The points close to the axis located $\pi/2$ radians from the scanning line, have little effect on the peaks and troughs of the response curve. They do have an effect over the region of frequencies nearest to their location. Their movement generally

raises or lowers that part of the curve where the slope is not very steep and does not change rapidly.

3. The effect of movement of critical points centrally located in the logarithmic plane is most difficult to predict and is most dependent on the particular distribution in question.

Two general statements can be now made. Unless there is great concentration or bunching of poles and zeros, then for each critical point there is a finite range of frequencies where the curve is critically affected by a small movement of the point. The movement is small enough so that it does not greatly affect the response outside this finite range. Furthermore, the value of the response can be kept constant at any frequency even if all poles and zeros are moved, provided that the distances to these critical points from the frequency in question is kept constant.

Keeping these facts in mind, it becomes a matter of experience to obtain the skill needed for making the perturbations so as to obtain rapid convergence.

XI. Synthesis of Driving-Point Functions

A necessary and sufficient condition for the physical realizability of a driving-point immittance is that it be a positive real function. This condition is guaranteed if the approximation is carried out in the real part plane (to be described below), providing certain restrictions on the positions of the critical points are satisfied.

The input immittance may be written as

$$F(p) = \frac{a_0 + a_1 p + \dots + a_i p^i}{b_0 + b_1 p + \dots + b_j p^j} \quad (17)$$

Along $j\omega$ this becomes

$$F(\omega) = \frac{m_1 + j\omega n_1}{m_2 + j\omega n_2} \quad (18)$$

where m and n are polynomials even in ω . Then,

$$\text{Re } F(\omega) = \frac{m_1 m_2 + \omega^2 n_1 n_2}{m_2^2 + \omega^2 n_2^2} \quad (19)$$

is an even function of ω . Substituting λ for ω gives

$$\operatorname{Re} F(\lambda) = \frac{A_0 + A_2 \lambda^2 + \dots + A_{21} \lambda^{21}}{B_0 + B_2 \lambda^2 + \dots + B_{2j} \lambda^{2j}} \quad (20)$$

Consider λ to be a complex variable

$$\lambda = \omega + j\omega \quad (21)$$

and express $\operatorname{Re} F(\lambda)$ in terms of its critical points

$$\operatorname{Re} F(\lambda) = H \frac{\pi(\lambda^2 - \lambda_0^2)}{\pi(\lambda^2 - \lambda_p^2)} \quad (22)$$

where λ_0 and λ_p are the zeros and poles, respectively. Factoring

$$\operatorname{Re} F(\lambda) = H \frac{\pi(\lambda - \lambda_0)(\lambda + \lambda_0)}{\pi(\lambda - \lambda_p)(\lambda + \lambda_p)} \quad (23)$$

But the coefficients of Eq. (20) are real so that the poles and zeros of the function are complex conjugate or real. Then terms containing the conjugate critical points of Eq. (23) must appear whenever these points are complex, or

$$\operatorname{Re} F(\lambda) = H \frac{(\lambda - \lambda_0)(\lambda - \lambda_0^*)(\lambda + \lambda_0)(\lambda + \lambda_0^*)}{(\lambda - \lambda_p)(\lambda - \lambda_p^*)(\lambda + \lambda_p)(\lambda + \lambda_p^*)} \quad (24)$$

Thus, it is seen that the poles and zeros are symmetrically positioned with respect to the real and imaginary axis in the λ -plane. In order that the real part be associated with a physical immittance function, there are two restrictions made on the locations of the poles and zeros.

1. There may be no poles anywhere on the real λ axis over the whole frequency spectrum.

2. Zeros on the real axis must be of even multiplicity. Critical points may appear anywhere else provided that the symmetry conditions are satisfied.

Symmetry with respect to both the real and the imaginary axis enables the analog to represent the whole function with any quadrant of the λ -plane, as

explained in Chapter IV. To assure physical realizability, the approximation is made to the real part of the required function. The necessary restrictions on the locations of the poles and zeros are very easily met in the analog. No pole-currents are introduced on the scanning axis and zero-currents on ω are even multiples of unit current. In order that no pole appear at infinity, no pole-currents may be introduced at the infinity return equipotential. This means that the number of zeros in the infinite region must be at least equal to the number of poles.

Simple analytic methods for determining $F(p)$ as a minimum reactance function are known (Reference 9) once the equation of $\text{Re } F(\lambda)$ has been found.

XII. The Experimental Results

The input admittance of a short circuited transmission line was synthesized over a specified frequency range with a lumped-parameter network by approximating to the input conductance.

The input admittance of a short circuited transmission line is given by

$$Y(\omega) = \frac{G + j\omega C}{R + j\omega L} \coth \ell \sqrt{(R - j\omega L)(G - j\omega C)} \quad (25)$$

where R , G , C , and L are the resistance, conductance, capacitance, and inductance per unit length. By neglecting the conductance, introducing

$$Z_c = \sqrt{L/C} \quad \text{lossless characteristic impedance}$$

$$\ell = R\ell/Z_c \quad \text{attenuation parameter}$$

$$\gamma = 2\pi\ell/\lambda = \omega\ell/v \quad \text{electrical argument}$$

$$v = 1/\sqrt{LC} \quad \text{velocity of propagation}$$

normalizing with respect to Z_c and substituting s for $j\gamma$, Eq. (25) is converted into

$$Y(s) = \frac{\coth s \sqrt{1 + \ell/s}}{\sqrt{(1 + \ell/s)}} \quad (26)$$

The Mittag-Leffler's partial fractions expansion of $Y(s)$ is (Reference 11)

$$Y(s) = \frac{1}{\rho + s} + \sum_{n=1}^{\infty} \frac{2s}{s^2 + \rho s + n^2 \pi^2} \quad (27)$$

A. Approximation for the Range $\tau \leq 3.0$

The approximation in this range was achieved by means of an economization process starting with the first two terms of the partial fractions expansion. The starting approximating function is

$$Y_1(s) = \frac{1}{\rho + s} + \frac{2s}{s^2 + \rho s + \pi^2} \quad (28)$$

for real frequencies this becomes,

$$Y_1(\tau) = \frac{1.085 (9.87 - 3\tau^2 + j 1.34\tau)}{4.41 - .894\tau^2 + j\tau(10.07 - \tau^2)} \quad \text{for } \rho = .447 \quad (29)$$

where the multiplicative constant 1.085 was chosen in the analog so as to make $G_1(\tau)$ an equal ripple approximation to $\text{Re } Y(s)$ for $\tau \leq 3.0$.

$$\text{Re } Y_1(\tau) = G_1(\tau) = \frac{1.085 (1.34\tau^4 - 8.5\tau^2 + 43.5)}{(4.41 - .894\tau^2)^2 + \tau^2 (10.07 - \tau^2)^2} \quad (30)$$

Substituting λ for τ and letting λ take complex values, the critical points of $G_1(\lambda)$ located in the first quadrant are:

$$\begin{aligned} \text{zeros: } \lambda_2 &= 2.39 \mid 25^\circ \\ \text{poles: } \lambda_1 &= .447 \mid 90^\circ \\ \lambda_3 &= 3.14 \mid 4.08^\circ \end{aligned} \quad (31)$$

These, then, were the original approximating critical points. The resultant analog function is to approximate $\text{Re } Y(s)$ for $\gamma \leq 3.0$ in an equal ripple manner. The curve to which the approximation was made is a plot of the real part of the function $Y(s)$ of Eq. (26). In this synthesis, the economization led to a very convenient form. It was found that the approximation to the real part could be made with a function having poles in the finite region and with all the zeros moved to infinity. This configuration of singularities is advantageous since the resulting network can always be synthesized with a reactive ladder structure containing no coupled coils and terminated in a pure resistance.

Consider Fig. 1.

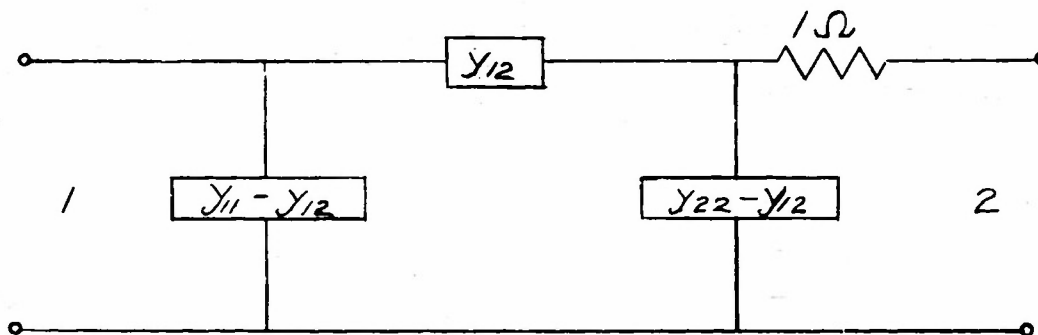


FIGURE 1

The short circuit transfer admittance is

$$\begin{aligned}
 Y_{12} &= \frac{y_{12}}{y_{22} + 1} \\
 &= \frac{N}{m_2 + p n_2} \\
 &= \frac{N/p n_2}{m_2/p n_2 + 1}
 \end{aligned} \tag{32}$$

then,

$$y_{12} = \frac{N}{p n_2} \tag{33}$$

and,

$$y_{22} = \frac{m_2}{pn_2} \quad \text{is purely reactive.} \quad (34)$$

Now, the zeros of power transfer are also the zeros of Y_{12} since a voltage V at the input terminals 1, results in an input power of

$$|V|^2 G_{in} = |VY_{12}|^2 \times 1 \quad (35)$$

when terminals 2 are shorted. Or,

$$G_{in} = |Y_{12}|^2 \quad (36)$$

If all the zeros of G_{in} are at infinity, then N is a constant.

Darlington synthesis (Reference 13) leads to an expansion of y_{22} in a continued fraction such that the zeros of y_{12} are all at infinity. The result is the required network, within a multiplicative constant, in the form of a reactive four-pole terminated in a resistor.

The results of the approximation are shown in Figs. MRI-13299 and MRI-13300 for G and Y , respectively. It is noted that two terms of the partial fractions expansion (with the proper multiplicative constant) is a good approximation to the required response. Removal of the zero, without perturbation of the poles results in a very large deviation of G_1 in the region $2.5 < \omega < 3.0$. The final approximation with the analog computer generally gives better results than the partial fractions expansion in the given range and the resulting network contains fewer elements. The networks arising from the two approximations are given in Fig. MRI-13301, the equation of the analog approximation being

$$Y_2(s) = \frac{s^2 + .62s + 3.14}{.364s^3 + .226s^2 + 3.48s + 1.45} \quad (37)$$

A comparison of the critical points in the two approximations is given in Table 1. The response curves of log amplitude against log frequency, as seen on the cathode ray tube, are shown in Fig. MRI-13297.

B. Approximation for the Range $\tau \leq 6.0$

Having arrived at satisfactory results for the range $\tau \leq 3.0$, the next step was an extended range; the range of approximation was doubled. The addition of another term of the partial fractions expansion gave the starting point approximation function,

$$Y_3(s) = \frac{1}{\rho + s} + \frac{2s}{s^2 + \rho s + \pi^2} + \frac{2s}{s^2 + \rho s + 4\pi^2} \quad (38)$$

Choosing a multiplicative constant of 1.31 so that the poles of the three terms of $Y_3(s)$ give an equal ripple approximation to $\text{Re } Y(s)$ for $\tau \leq 6.0$, Eq. (38) becomes,

$$Y_3(s) = \frac{1.31 (5s^4 + 4.47s^3 + 149.1s^2 + 66.1s + 390)}{s^5 + 1.341s^4 + 49.97s^3 + 44.2s^2 + 399s + 174} \quad (39)$$

$$\rho = .447$$

The desired analog function is to approximate $\text{Re } Y(s)$ for $\tau \leq 6.0$ in an equal ripple manner with all zeros in the real part plane located at infinity. The poles of $\text{Re } Y_3(s)$ give the unperturbed location of the singularities. These are obtainable from the solutions of the equations

$$\begin{aligned} \rho + j\lambda &= 0 \\ \pi^2 - \lambda^2 + j\rho\lambda &= 0 \\ 4\pi^2 - \lambda^2 + j\rho\lambda &= 0 \end{aligned} \quad (40)$$

From which the poles in the first quadrant of λ are:

$$\begin{aligned} \lambda_1 &= .447 \quad | \quad 90^\circ \\ \lambda_3 &= 3.14 \quad | \quad 4.08^\circ \\ \lambda_5 &= 6.28 \quad | \quad 2.04^\circ \end{aligned} \quad (41)$$

The results of the analog approximation are shown in Figs. MRI-13302 and MRI-13303 for G and Y , respectively. Fig. MRI-13302 clearly shows that moving the zeros of the partial fractions approximation to infinity results in

large deviations from the required response. Adjustments of pole positions, however, result in an analog approximation which is considerably better. Fig. MRI-13303 gives the resultant plots of the amplitude of the admittance. Although the analog approximation deviates from the exact amplitude of the admittance to a marked degree in the vicinity of the peak (at $\omega = 3.14$), it is considerably better than the partial fractions expansion. Furthermore, an economization of two elements has been effected, and the analog network is of simpler configuration. The networks arising from the two approximations are given in Fig. MRI-13304, the equation of the analog approximation being

$$Y_4(s) = \frac{s^4 + .878s^3 + 36s^2 + 24.7s + 76.4}{.191s^5 + .167s^4 + 8.94s^3 + 6.54s^2 + 68.4s + 33.2} \quad (42)$$

It should be emphasized that the analog approximation is made to the real part, and the resultant approximation to the amplitude is computed by constructing the complete complex admittance according to References 9 and 13.

A comparison of the critical points in the two approximations for the extended range is given in Table 2. It is to be observed that relatively small adjustments were required to obtain the improvements. The response curves of log amplitude against log frequency, as seen on the cathode ray tube, are shown in Fig. MRI-13298.

XIII. Conclusion

An electrostatic analog computer, capable of approximating to the real or imaginary part as well as to the amplitude, has been successfully built. The machine gives results well within experimental accuracy and is a valuable laboratory tool. Although the method is one of trial-and-error, the amount of computational work, as compared to that normally encountered in the analytical approach to the approximation problem, is greatly reduced.

The perturbation technique, as well as the proper use of the logarithmic transformation, simplifies the approach. The continuous scan feature of the analog makes it possible to immediately observe changes in the approximation and solutions are obtainable in a very short time. The approach from the real part leads to realizable driving-point functions, while economization reduces the number of elements. The experimental results show that relatively small adjustments in the positions of the critical points effected substantial improvements. It should be pointed out that the same approximations to the transmission line could have been gotten without a knowledge of the partial fractions approximating networks since the location of critical points are readily obtainable.

	ANALOG APPROXIMATION		PARTIAL FRACTIONS APPROX.	
	G_2 λ -PLANE	Y_2 S-PLANE	G_1 λ -PLANE	Y_1 S-PLANE
POLES	$.42/90^\circ$	$.42/180^\circ$	$.447/90^\circ$	$.447/180^\circ$
	$3.08/1.86^\circ$	$3.08/91.86^\circ$	$3.14/4.08^\circ$	$3.14/94.08^\circ$
ZEROS	AT ∞	$1.8/99.94^\circ$	$2.39/25^\circ$	$1.81/97.07^\circ$

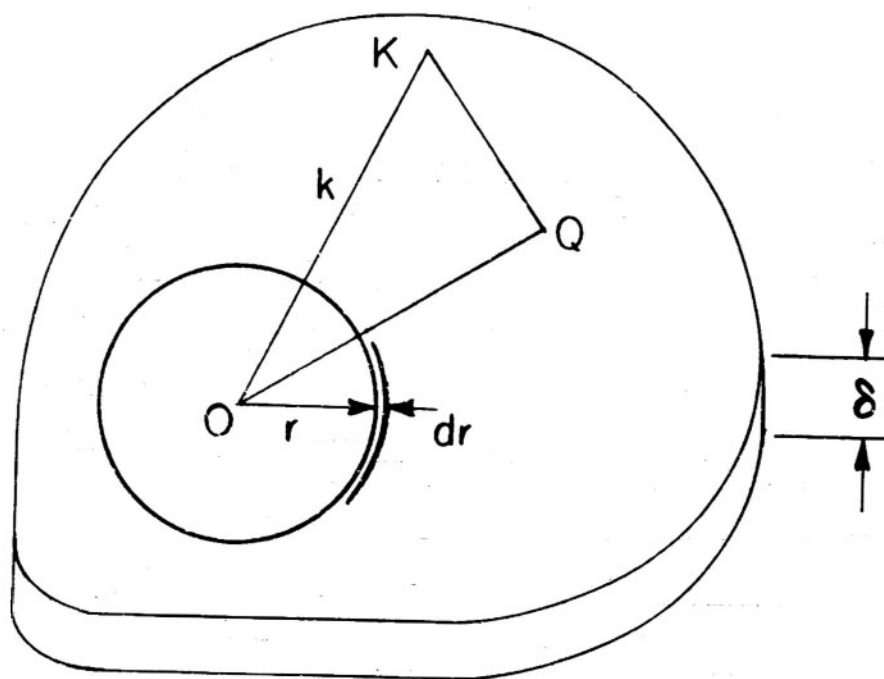
TABLE 1

	ANALOG APPROXIMATION		PARTIAL FRACTIONS APPROX.	
	G_4 λ -PLANE	Y_4 S-PLANE	G_3 λ -PLANE	Y_3 S-PLANE
POLES	$.48/90^\circ$	$.48/180^\circ$	$.447/90^\circ$	$.447/180^\circ$
	$3.12/2.64^\circ$	$3.12/92.64^\circ$	$3.14/4.08^\circ$	$3.14/94.08^\circ$
	$6.10/.53^\circ$	$6.10/90.53^\circ$	$6.28/2.04^\circ$	$6.28/92.04^\circ$
ZEROS	AT ∞	$1.48/92.96^\circ$ $5.79/93.6^\circ$	—	$1.29/100.38^\circ$ $5.59/92.2^\circ$

TABLE 2

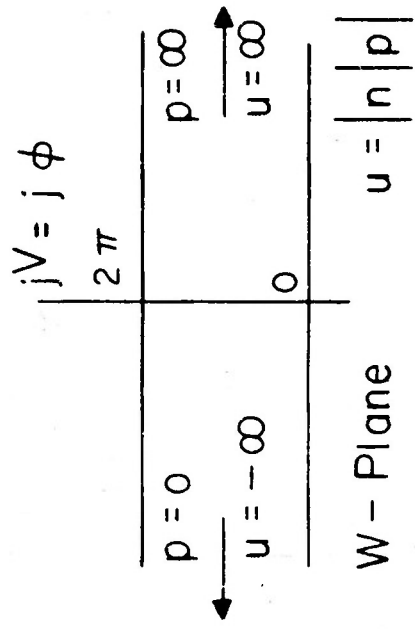
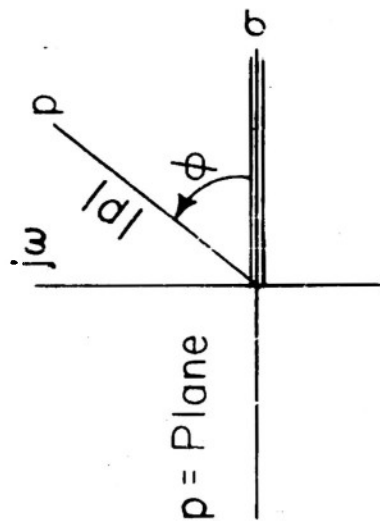
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THE CONDUCTING PLANE

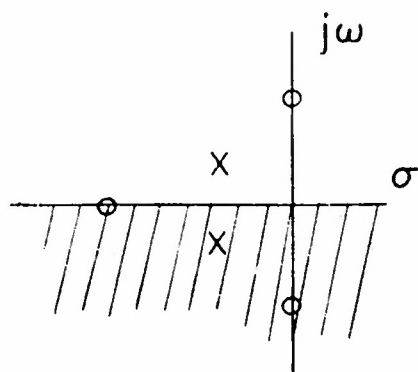
FIGURE 1



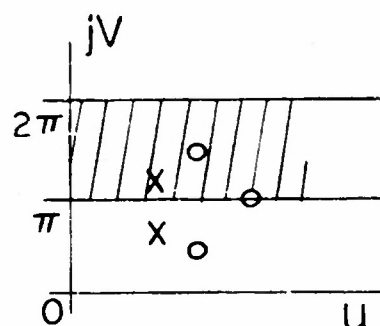
$$W = \ln p$$

LOGARITHMIC TRANSFORMATION

FIGURE 2

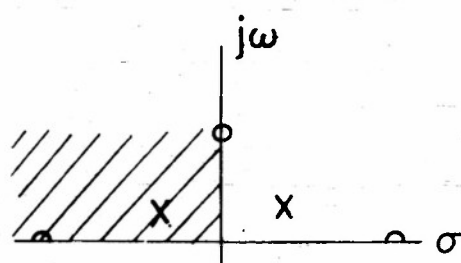


P = Plane

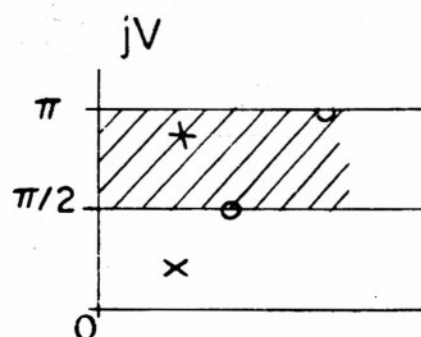


P = Plane

(A) SYMMETRY ABOUT REAL AXIS



P = Plane



W = Plane

(B) SYMMETRY ABOUT IMAGINARY AXIS

FIGURE 3

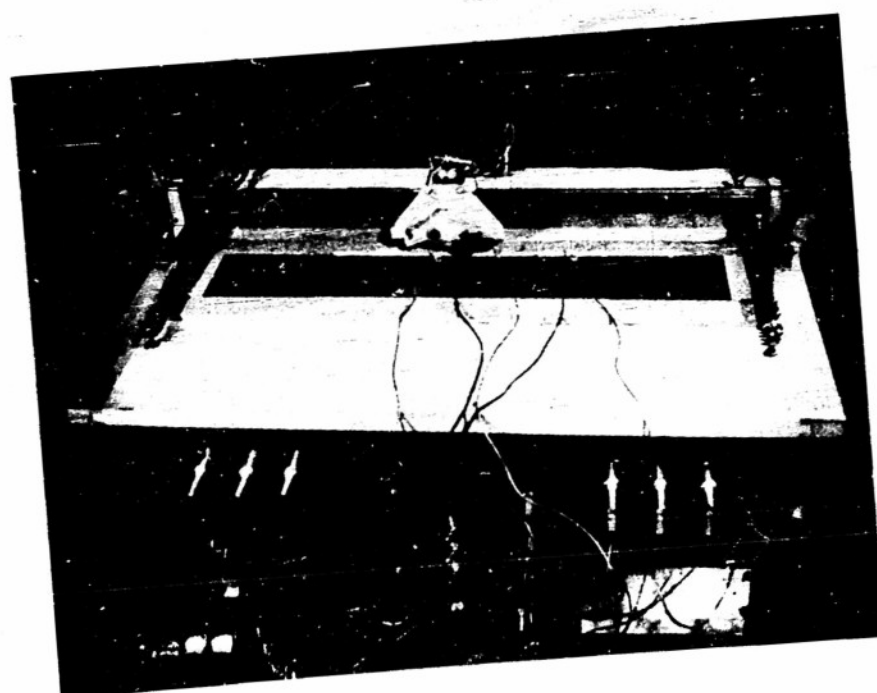
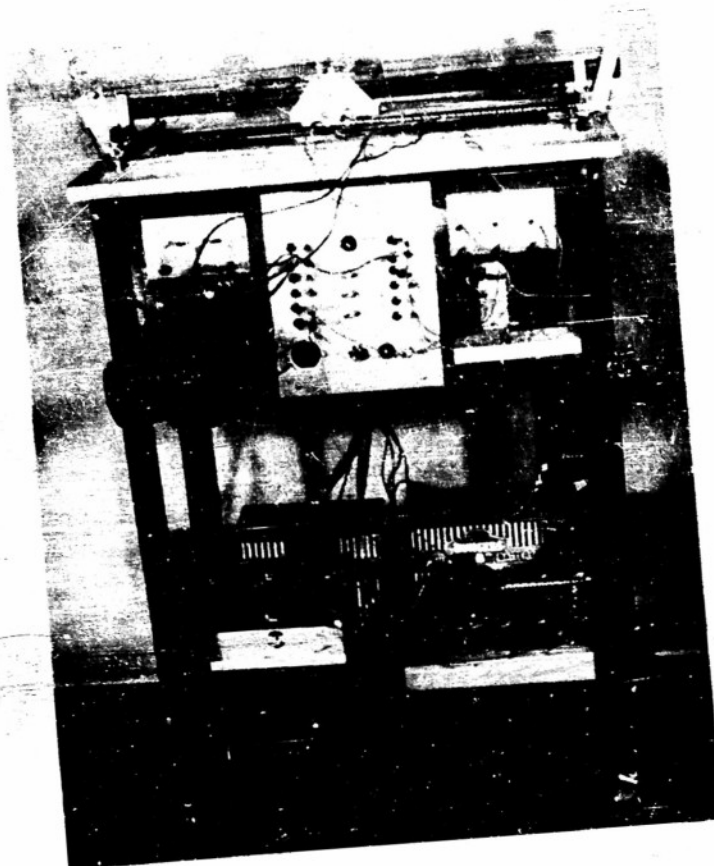


FIG. 4 THE ANALOG COMPUTER

MRI 13296

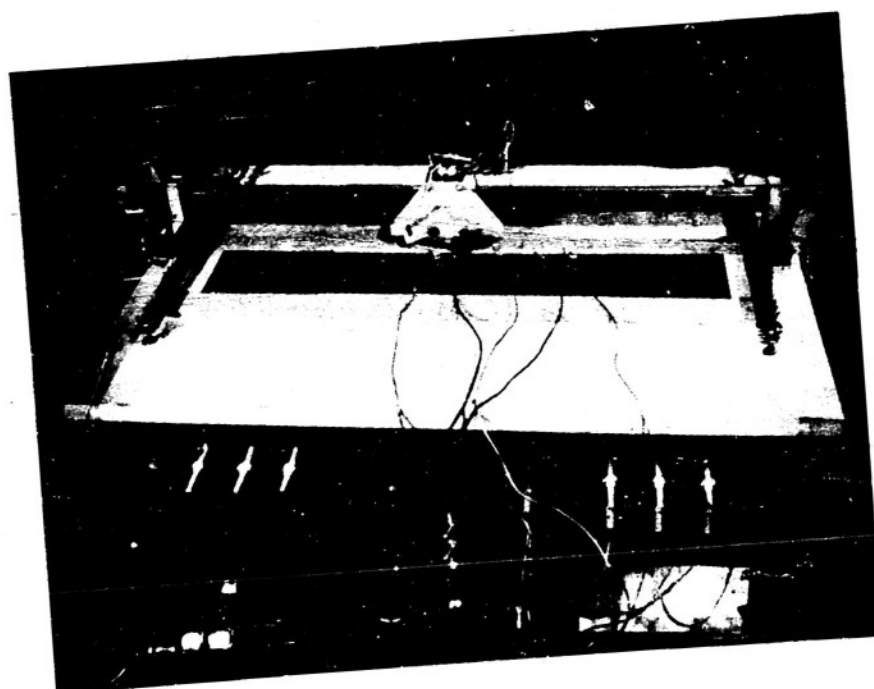
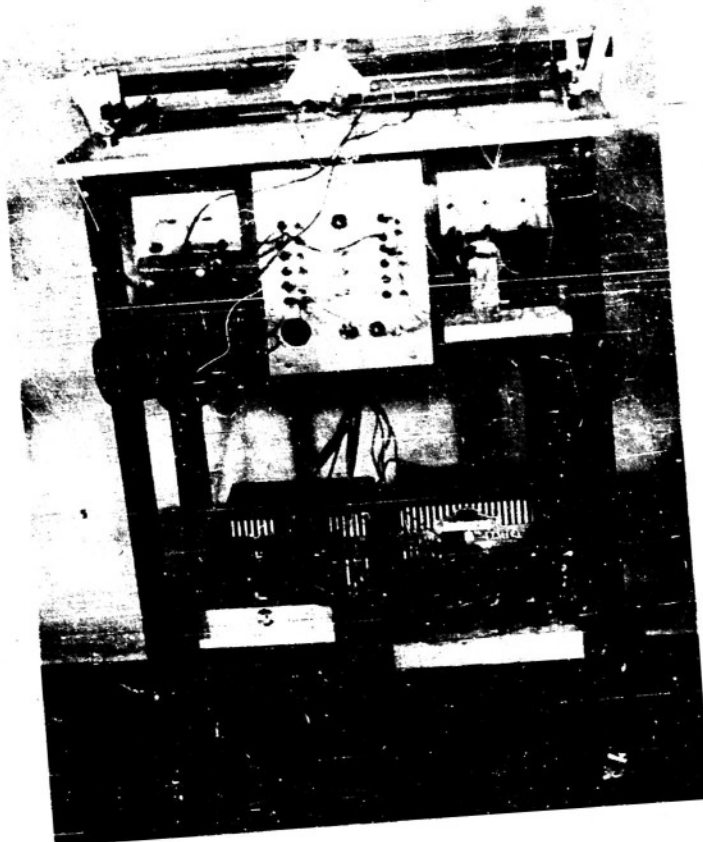
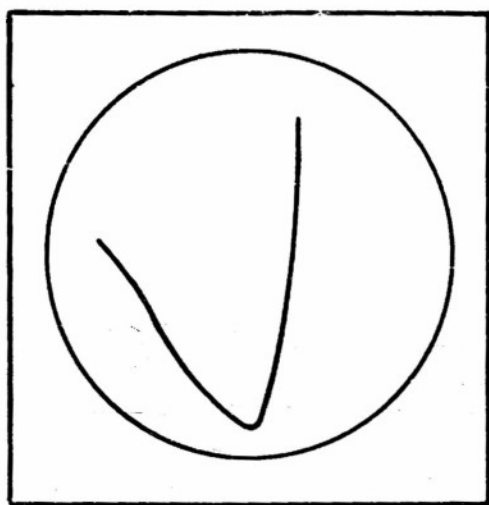
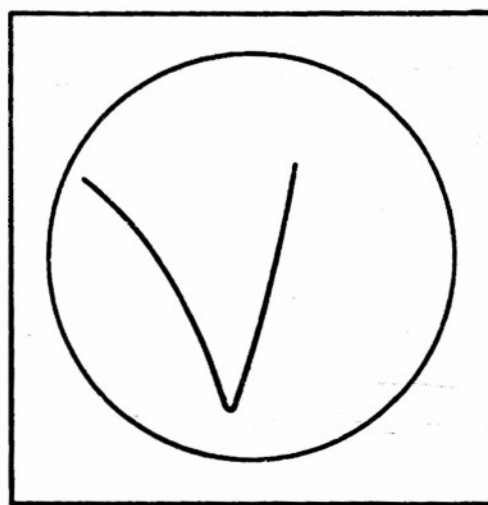


FIG. 4 THE ANALOG COMPUTER

MRI 13296

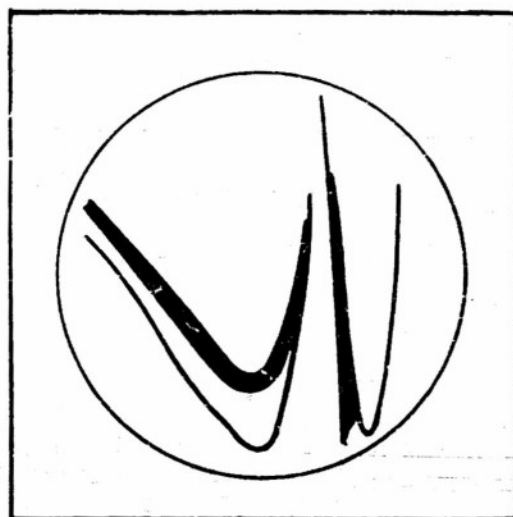


A. $G(t)$

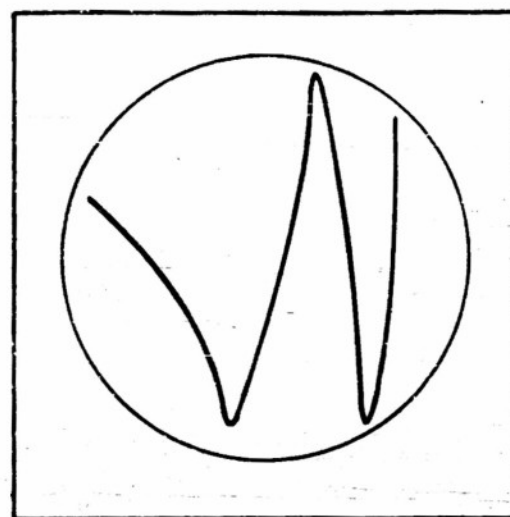


B. $\gamma(t)$

FIG. 6 APPROXIMATION FOR $T \leq 3.0$

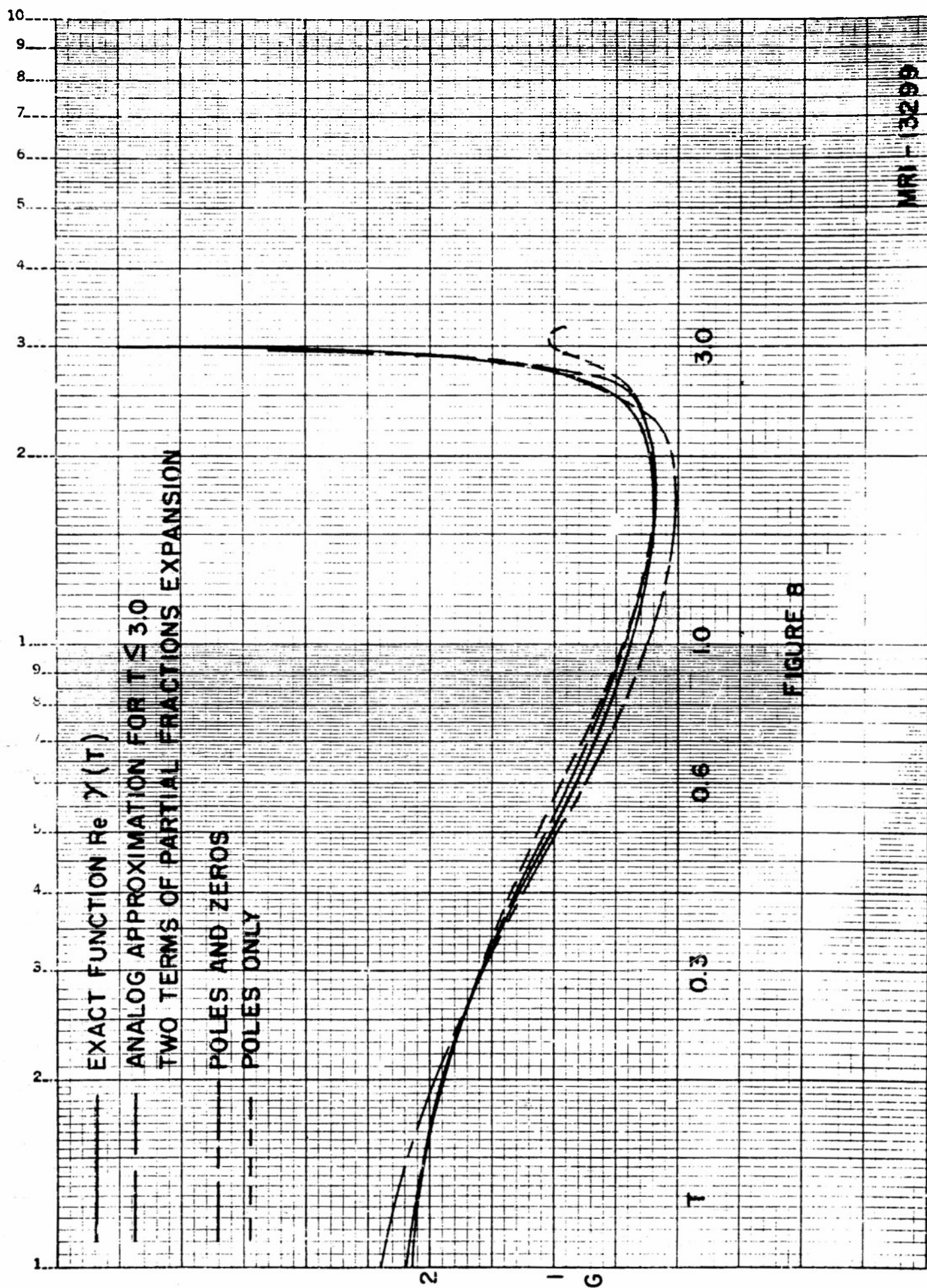


A. $G(t)$



B. $\gamma(t)$

FIG.7 APPROXIMATION FOR $T \leq 6.0$



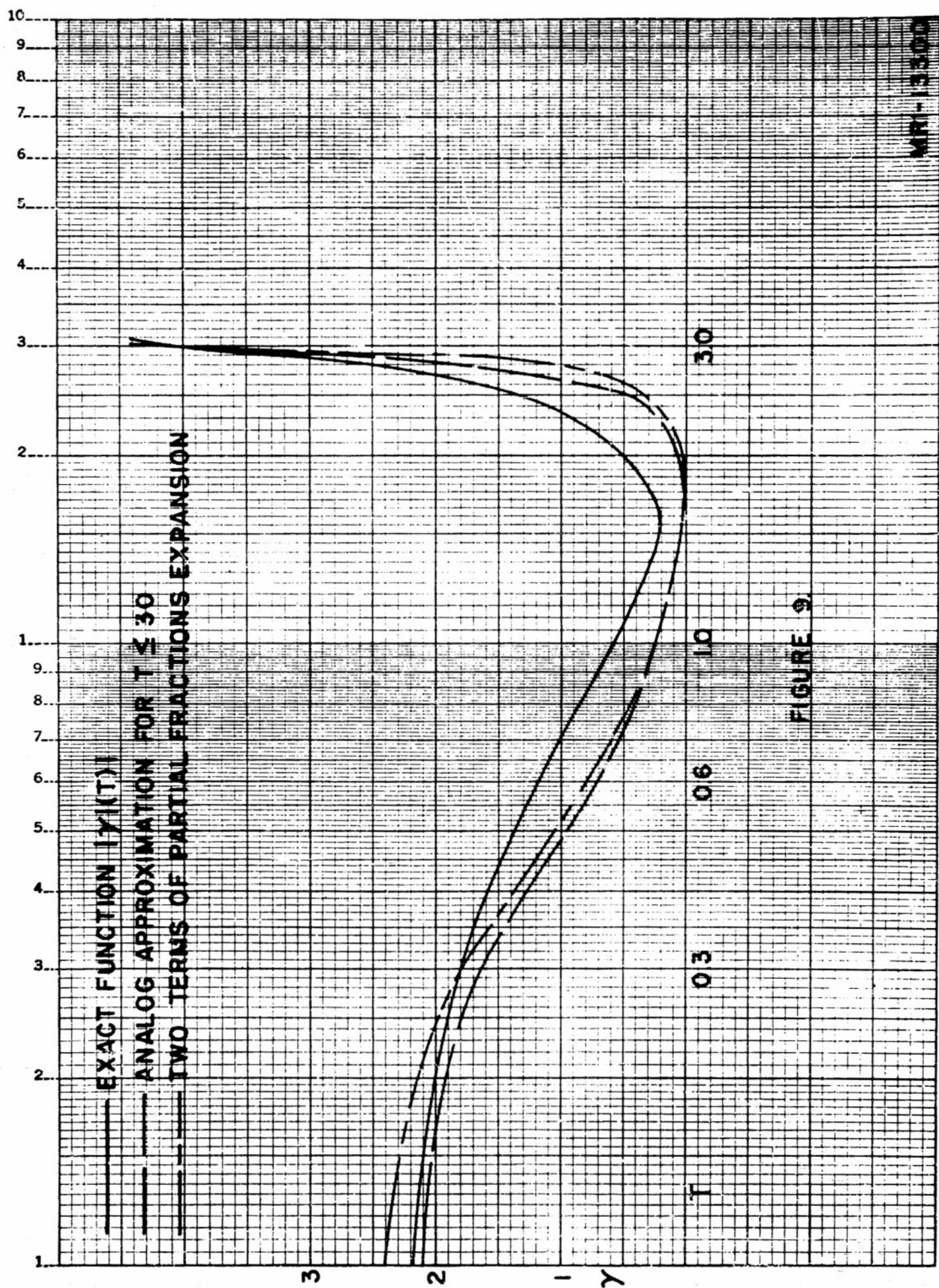
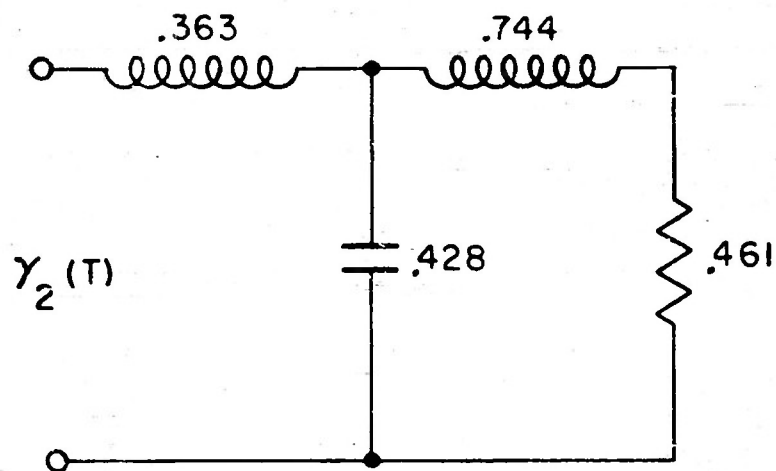
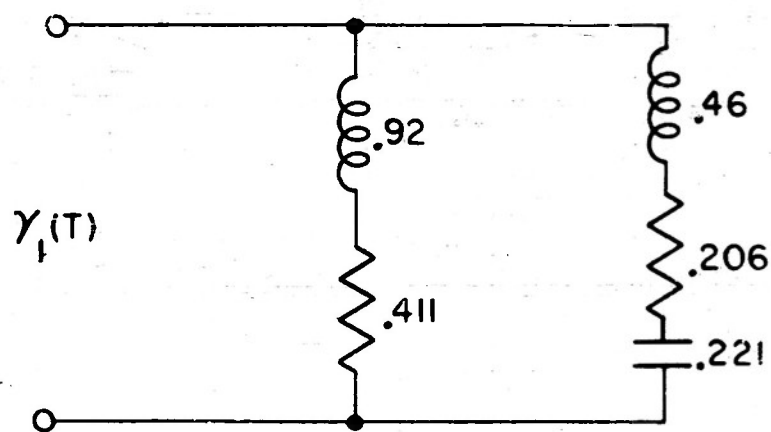


FIGURE 9

MR-13500



A ANALOG APPROXIMATION



B PARTIAL FRACTION APPROXIMATION

$T \leq 30$

FIGURE 10

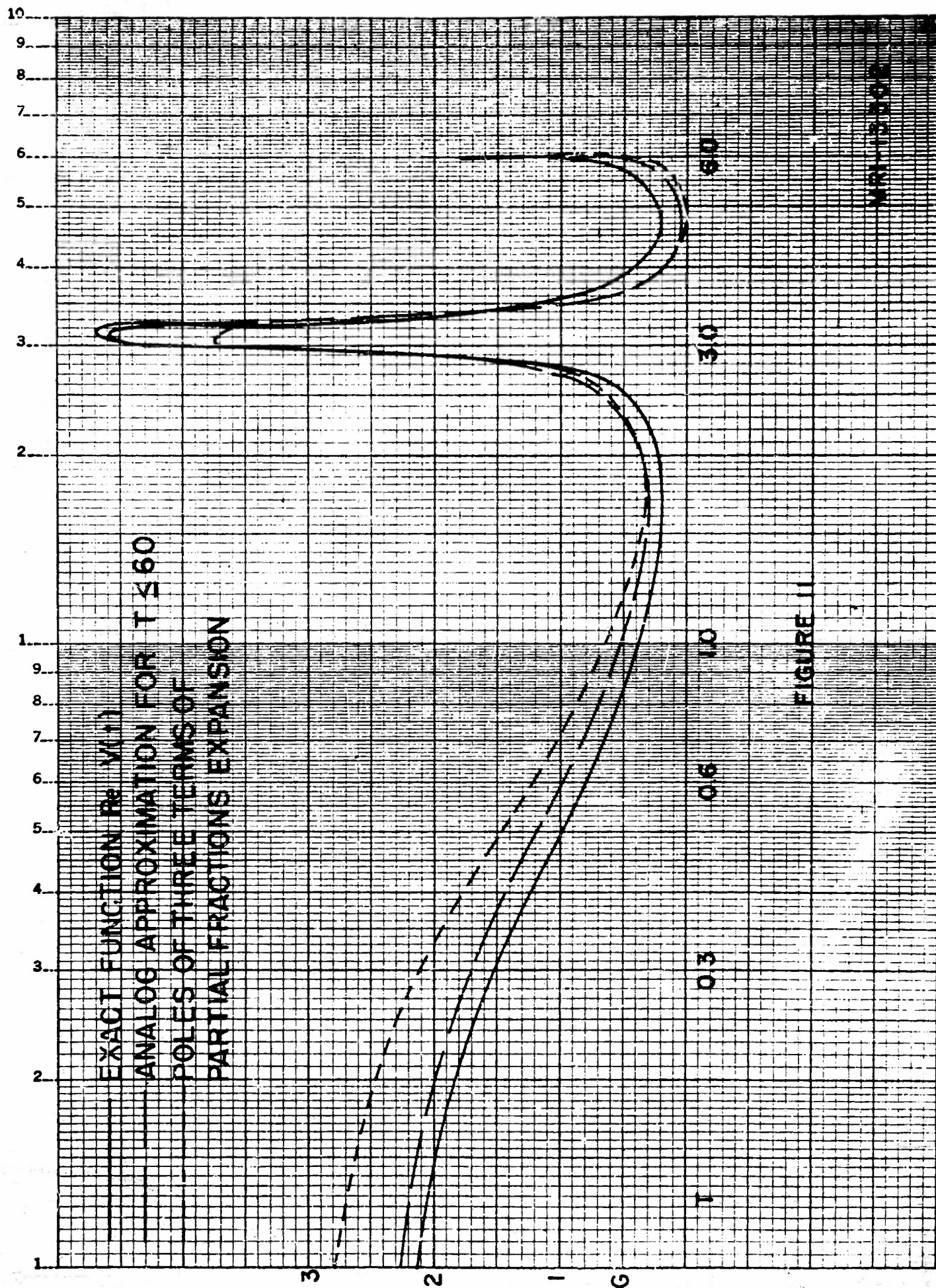


FIGURE 11

W81-13308

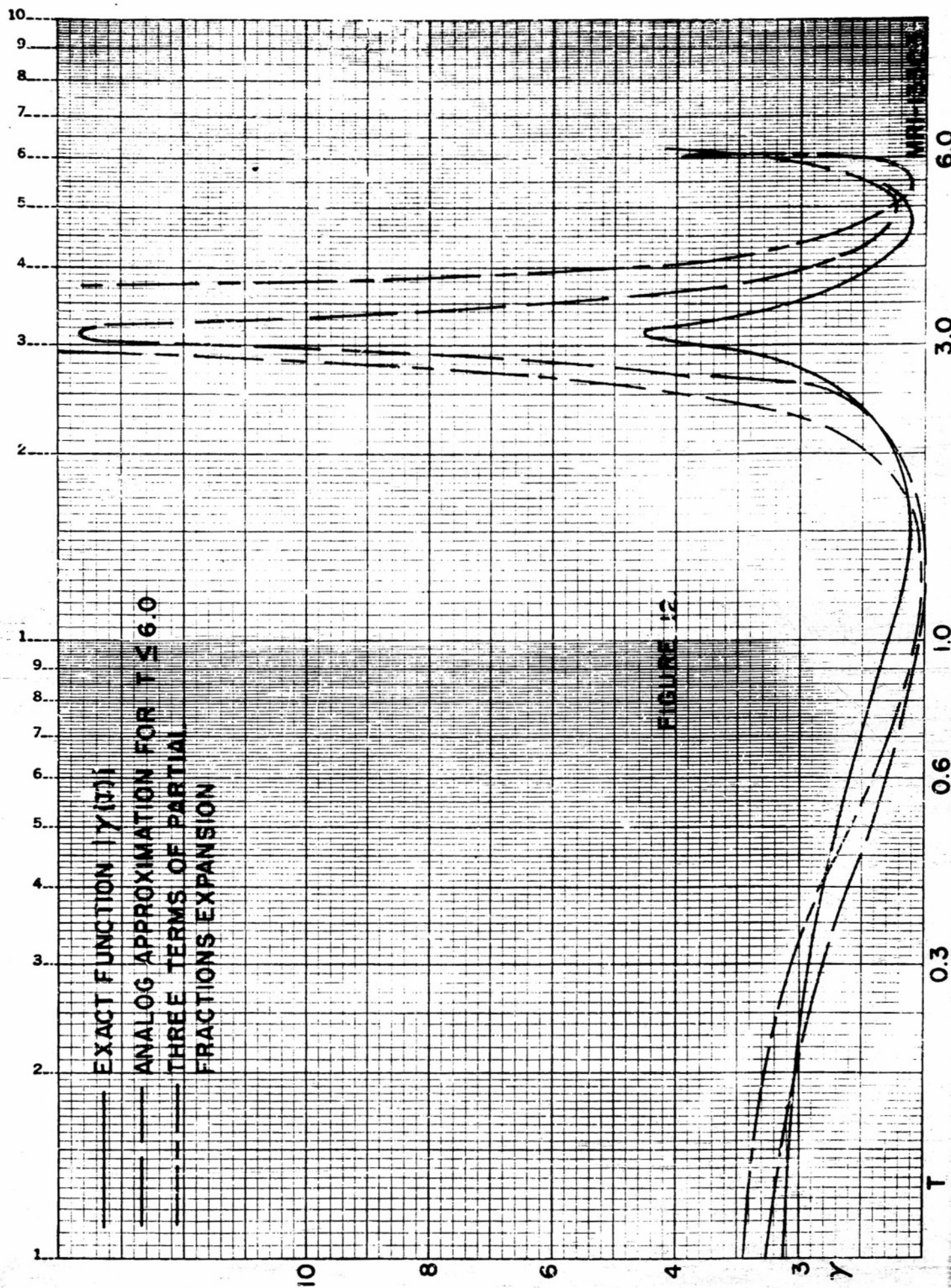
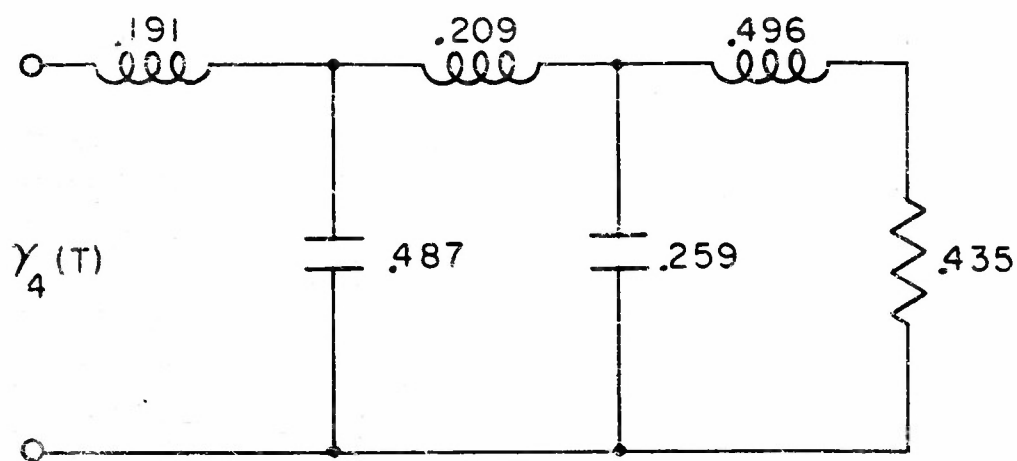
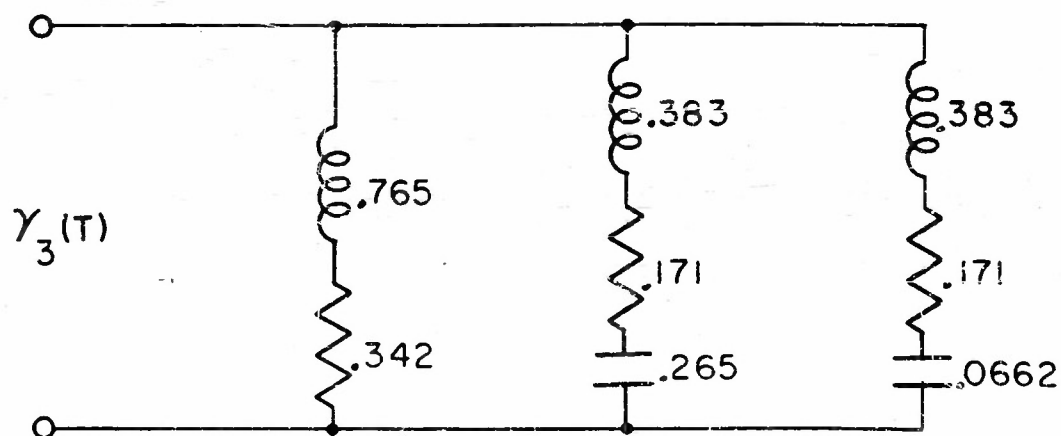


FIGURE 12



A. ANALOG APPROXIMATION



B. PARTIAL FRACTIONS APPROXIMATION

$$T \leq 60$$

FIGURE 13